

OUTP-00-25P

hep-th/0006030

June 2000

Multi-Brane Worlds and modification of gravity at large scales.

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Abstract

We discuss the implications of multi-brane constructions involving combinations of positive and negative tension brane and show how anomalously light KK states emerge when negative tension “-” branes are sandwiched between “+” branes. We present a detailed study of a “+ - - +” brane assignment which interpolates between two models that have been previously proposed in which gravity is modified at large scales due to the anomalously light states. We show that it has the peculiar characteristic that gravity changes from four dimensional (4D) to 5D at large distances and returns to 4D at even larger scales. We also consider a crystalline universe which leads to a similar structure for gravity. The problems associated with intermediate negative tension branes are discussed and a possible resolution suggested.

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1 Introduction

The conjecture that we may live on a brane in a space-time with more than four dimensions is quite old [1] but has been the subject of renewed interest in recent years with the realisation that such structures are common in string theories. The models of Antoniadis, Arkani-Hamed, Dimopoulos, Dvali (AADD) [2] and of Randall, Sundrum (RS) [3, 4] have the common characteristic that the Standard Model (SM) fields are localized on a 3-brane and gravity propagates in all the space-time dimensions (the “bulk”). They can provide novel explanations to the Planck hierarchy problem based either on the size of the bulk volume (AADD), or on an exponential warp factor that enters the metric (RS). The RS construction in particular is very attractive because all the parameters of the model may be of the same magnitude while still generating a very large hierarchy.

The RS model consists of two 3-branes of opposite tension sitting at the fixed points of an S^1/Z_2 orbifold with AdS_5 bulk geometry. Gravity is localized on the brane of positive tension (“+” brane) whereas an exponential “warp” factor generates a scale hierarchy on the brane of negative tension (“−” brane). There are two possible physical interpretations of this model. Placing the Standard Model (SM) fields on the “−” brane [3] provides an explanation of the Planck hierarchy problem. Placing the SM fields on the “+” brane [4] allows the “−” brane to be put at infinity, *i.e.* decompactify the orbifold. In this case, however, gravity is localised on the “+” brane so the world still appears four dimensional (see also [5]). As far as the first possibility is concerned, there has been a lot of discussion [6] whether we can have acceptable theory of gravity and cosmology if we live on a “−” brane. To avoid this problem Lykken and Randall (LR) [7] proposed a “+ + −” brane configuration in which the SM fields reside on the intermediate positive tension brane (which has vanishingly small tension) and which still has a warp factor explanation of the mass hierarchy.

In reference [8] we considered an alternative “+ − +” configuration, again with the SM fields on a positive tension brane and a warp factor. This model (KMPRS) has a novel phenomenology [9] quite distinct from the RS and LR models. This is mainly due to the intermediate “−” brane leading to an anomalously light first Kaluza-Klein (KK) as discussed below. This feature gives rise to the exotic possibility of “bigravity”, in which the first KK state has a Compton wavelength of the order of 10^{26} cm corresponding to 1% of the size of the observable universe, while, at the same time, all the remaining states of the KK tower have Compton wavelengths below 1mm. In such a scenario the gravitational attraction as we feel it is the net effect of the exchange of the ordinary 4D graviton *and* the ultralight KK state. This leads to the prediction of modifications of gravity not only at the millimeter scale (due to the higher KK modes) but also the ultralarge scale (of $\mathcal{O}(10^{26}$ cm)!) Indeed, gravity becomes weaker at ultralarge distances due to the Yukawa suppression of the first

KK mode.

Independently and again in the context of the RS scenario, Gregory, Rubakov, Sibiryakov (GRS) have recently suggested [10] a construction in which gravity is also modified at both small and at ultralarge scales. The GRS model modifies the decompactified RS model[4] by adding a “−” brane of half the tension of the “+” brane and requiring flat space to the right of the new “−” brane. This model does not have a normalizable 4D graviton but generates 4D gravity at intermediate distances due to a resonance-like behaviour of the wavefunctions of the KK states continuum. Gravity in this picture appears to be “quasi-localized” and for ultra large scales becomes five dimensional. In this scenario gravity is also modified at short scales (but this time much shorter than the millimeter scale).

Although the KMPRS and GRS models look quite different, they share the key element of freely moving “−” branes which generate the tunneling effects responsible for the anomalously light states. It was shown in [11] that these two models are the limiting cases of a more general “+ − − +” multi-brane model that interpolates between the “bigravity” KMPRS model and the “quasi-localized” gravity GRS model. This model can be readily derived from the KMPRS model by cutting in half the intermediate “−” brane and considering flat space between the two new “−” branes. Moreover, in [11] it was conjectured that every construction having “−” branes between “+” branes is capable of giving a “multigravity” scenario where gravity is generated not only by the 4D graviton but also by a number (finite or infinite) of KK states. The crystal universe considered by [12] (see also [13]) is an interesting example⁵.

The purpose of this paper is threefold. We first we will consider the “+ + − −” LR construction but with the intermediate brane carrying non-negligible tension. We show that this model does not have an anomalously light KK state and hence cannot generate a “multigravity” scenario. Analysis of the reason for this suggests that “+” branes not separated by “−” branes do not lead to anomalously light modes. Secondly, we examine in detail the “+ − − +” “multigravity” model and show exactly how it interpolates between the KMPRS and the GRS model. Gravity in this case has the quite peculiar characteristic that it changes from 4D to 5D at ultra-large scales and then again to 4D at even larger scales. We also analyse the infinite crystalline universe and show how the band structure of the KK modes gives rise to “multigravity”. Finally we discuss the role of the radion and the question whether these theories give rise to viable theories.

⁵We disagree with the statements made in [12] about the absence of a zero mode band.

2 The three-brane "++-" Model

The "++-" model (see Fig.1) consists of three parallel 3-branes in AdS_5 spacetime with orbifold topology, two of which are located at the orbifold fixed points $L_0 = 0$ and L_2 while the third one is moving at distance L_1 in between. In order to get 4D flat space with this configuration, it turns out that the AdS_5 space must have different cosmological constants Λ_1 and Λ_2 between the first - second and the second - third brane respectively with $|\Lambda_2| > |\Lambda_1|$ (see [14] for constructions of different bulk cosmological constants). The action of the above setup (if we neglect the matter contribution on the branes in order to find a suitable vacuum solution) is given by:

$$S = \int d^4x \int_{-L_2}^{L_2} dy \sqrt{-G} [-\Lambda(y) + 2M^3 R] - \sum_i \int_{y=y_i} d^4x V_i \sqrt{-\hat{G}^{(i)}} \quad (1)$$

with $\Lambda(y) = \begin{cases} \Lambda_1 & , y \in [0, L_1] \\ \Lambda_2 & , y \in [L_1, L_2] \end{cases}$

where $\hat{G}_{\mu\nu}^{(i)}$ is the induced metric on the branes, V_i are their tensions and M the 5D fundamental scale. We consider as in [8] the vacuum metric ansatz that respects 4D Poincaré invariance:

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \quad (2)$$

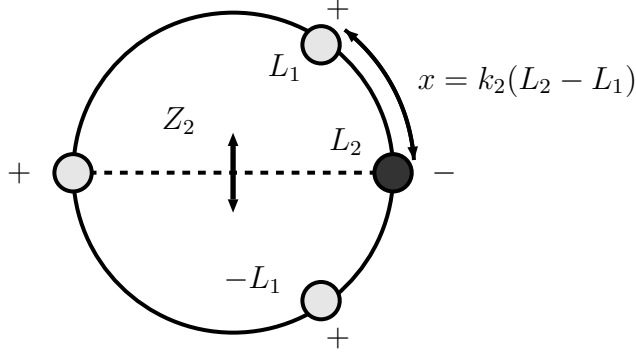


Figure 1: The brane locations in the three-brane "++-" model. The bulk curvature between the "++" branes is k_1 and between the "++" and "--" brane is k_2 .

The above ansatz substituted in the Einstein equations requires that $\sigma(y)$ satisfies the differential equations:

$$\sigma'' = \sum_i \frac{V_i}{12M^3} \delta(y - L_i) \quad \text{and} \quad (\sigma')^2 = \begin{cases} k_1^2 & , y \in [0, L_1] \\ k_2^2 & , y \in [L_1, L_2] \end{cases} \quad (3)$$

where $k_1 = \sqrt{\frac{-\Lambda_1}{24M^3}}$ and $k_2 = \sqrt{\frac{-\Lambda_2}{24M^3}}$ are effectively the bulk curvatures in the two regions between the branes. The solution of these equations for $y > 0$ it is given by:

$$\sigma(y) = \begin{cases} k_1 y & , y \in [0, L_1] \\ k_2(y - L_1) + k_1 L_1 & , y \in [L_1, L_2] \end{cases} \quad (4)$$

Einstein's equations impose the following conditions on the brane tensions :

$$V_0 = 24M^3 k_1, \quad V_1 = 24M^3 \frac{k_2 - k_1}{2}, \quad V_2 = -24M^3 k_2 \quad (5)$$

If we consider massless fluctuations of this vacuum metric and then integrate over the 5-th dimension, we find the 4D Planck mass is given by:

$$M_{\text{Pl}}^2 = M^3 \left[\frac{1}{k_1} (1 - e^{-2k_1 L_1}) + \frac{1}{k_2} e^{2(k_2 - k_1)L_1} (e^{-2k_2 L_1} - e^{-2k_2 L_2}) \right] \quad (6)$$

The above formula tells us that for large enough kL_1 and kL_2 the four mass scales M_{Pl} , M , k_1 and k_2 can be taken to be of the same order. Thus we take $k_1, k_2 \sim \mathcal{O}(M)$ in order not to introduce a new hierarchy, with the additional restriction $k_1 < k_2 < M$ so that the bulk curvature is small compared to the 5D Planck scale and we can trust the solution. Furthermore, if we put matter on the second brane all the physical masses m on it will be related to the mass parameters m_0 of the fundamental 5D theory by the conformal ‘‘warp’’ factor

$$m = e^{-\sigma(L_1)} m_0 = e^{-k_1 L_1} m_0 \quad (7)$$

This allows us to put the SM states on the intermediate ‘‘+’’ brane, solving the Planck hierarchy problem by choosing e^{-kL_1} to be of $\mathcal{O}(10^{-15})$, *i.e.* $L_1 \approx 35k^{-1}$.

The KK spectrum can be as usual found by considering the linear ‘‘massive’’ fluctuations of the metric. We use the following convention⁶:

$$ds^2 = \left[e^{-2\sigma(y)} \eta_{\mu\nu} + \frac{2}{M^{3/2}} h_{\mu\nu}(x, y) \right] dx^\mu dx^\nu + dy^2 \quad (8)$$

We expand the field $h_{\mu\nu}(x, y)$ in terms of the 4D graviton zero mode and the KK states plane waves $h_{\mu\nu}^{(n)}(x)$:

$$h_{\mu\nu}(x, y) = \sum_{n=0}^{\infty} h_{\mu\nu}^{(n)}(x) \Psi^{(n)}(y) \quad (9)$$

where $(\partial_\kappa \partial^\kappa - m_n^2) h_{\mu\nu}^{(n)} = 0$ after fixing the gauge as $\partial^\alpha h_{\alpha\beta}^{(n)} = h^{(n)\alpha}_\alpha = 0$. The wavefunction $\Psi^{(n)}(y)$ obeys a second order differential equation and carries all the information about the

⁶Here we have ignored the radion modes that could be used to stabilize the brane positions L_1 and L_2 . For discussion and possible stabilization mechanisms see [15]

effect of the non-factorizable geometry on the graviton and the KK states. After a change of coordinates and a redefinition of the wavefunction the problem reduces to the solution of an ordinary Schrödinger equation:

$$\left\{ -\frac{1}{2}\partial_z^2 + V(z) \right\} \hat{\Psi}^{(n)}(z) = \frac{m_n^2}{2} \hat{\Psi}^{(n)}(z) \quad (10)$$

where the potential $V(z)$ for $z > 0$ has the form:

$$\begin{aligned} V(z) = & \frac{15}{8[g(z)]^2} [k_1^2(\theta(z) - \theta(z - z_1)) + k_2^2(\theta(z - z_1) - \theta(z - z_2))] \\ & - \frac{3}{2g(z)} \left[k_1\delta(z) + \frac{(k_2 - k_1)}{2}\delta(z - z_1) - k_2\delta(z - z_2) \right] \end{aligned} \quad (11)$$

The new coordinates and wavefunction in the above equations are defined by:

$$z \equiv \begin{cases} \frac{e^{k_1 y} - 1}{k_1} & , y \in [0, L_1] \\ \frac{e^{k_2(y - L_1)} + k_1 L_1}{k_2} + \frac{e^{k_1 L_1} - 1}{k_1} - \frac{e^{k_1 L_1}}{k_2} & , y \in [L_1, L_2] \end{cases} \quad (12)$$

$$\hat{\Psi}^{(n)}(z) \equiv \Psi^{(n)}(y)e^{\sigma/2} \quad (13)$$

with the symmetric z for $y < 0$ and the function $g(z)$ as:

$$g(z) = \begin{cases} k_1 z + 1 & , z \in [0, z_1] \\ k_2(z - z_1) + k_1 z_1 + 1 & , z \in [z_1, z_2] \end{cases} \quad (14)$$

where $z_1 = z(L_1)$ and $z_2 = z(L_2)$. The change of coordinates has been chosen so that there are no first derivative terms in the differential equation. Furthermore, in this coordinate system it can be easily seen that the vacuum metric takes the conformally flat form:

$$ds^2 = \frac{1}{g(z)^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) \quad (15)$$

The potential (11) always gives rise to a massless graviton zero mode which reflects the fact that Lorentz invariance is preserved in 4D spacetime. Its wavefunction is given by:

$$\hat{\Psi}^{(0)} = \frac{A}{[g(z)]^{3/2}} \quad (16)$$

with normalization factor A determined by the requirement $\int_{-z_2}^{z_2} dz [\hat{\Psi}^{(0)}(z)]^2 = 1$, chosen so that we get the standard form of the Fierz-Pauli Lagrangian (the same holds for the normalization of all the other states).

For the massive KK modes the solution is given in terms of Bessel functions. For y lying in the regions $\mathbf{A} \equiv [0, L_1]$ and $\mathbf{B} \equiv [L_1, L_2]$, we have:

$$\hat{\Psi}^{(n)} \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \end{Bmatrix} = N_n \begin{Bmatrix} \sqrt{\frac{g(z)}{k_1}} \left[Y_2 \left(\frac{m_n}{k_1} g(z) \right) + A J_2 \left(\frac{m_n}{k_1} g(z) \right) \right] \\ \sqrt{\frac{g(z)}{k_2}} \left[B_1 Y_2 \left(\frac{m_n}{k_2} g(z) \right) + B_2 J_2 \left(\frac{m_n}{k_2} g(z) \right) \right] \end{Bmatrix} \quad (17)$$

The boundary conditions (one for the continuity of the wavefunction at z_1 and three for the discontinuity of its first derivative at 0, z_1 , z_2) result in a 4×4 homogeneous linear system which, in order to have a non-trivial solution, should have a vanishing determinant:

$$\begin{vmatrix} Y_1 \left(\frac{m}{k_1} \right) & J_1 \left(\frac{m}{k_1} \right) & 0 & 0 \\ 0 & 0 & Y_1 \left(\frac{m}{k_2} g(z_2) \right) & J_1 \left(\frac{m}{k_2} g(z_2) \right) \\ Y_1 \left(\frac{m}{k_1} g(z_1) \right) & J_1 \left(\frac{m}{k_1} g(z_1) \right) & -\sqrt{\frac{k_1}{k_2}} Y_1 \left(\frac{m}{k_2} g(z_1) \right) & -\sqrt{\frac{k_1}{k_2}} J_1 \left(\frac{m}{k_2} g(z_1) \right) \\ Y_2 \left(\frac{m}{k_1} g(z_1) \right) & J_2 \left(\frac{m}{k_1} g(z_1) \right) & -\sqrt{\frac{k_1}{k_2}} Y_2 \left(\frac{m}{k_2} g(z_1) \right) & -\sqrt{\frac{k_1}{k_2}} J_2 \left(\frac{m}{k_2} g(z_1) \right) \end{vmatrix} = 0 \quad (18)$$

(The subscript n on the masses m_n has been suppressed.)

2.1 Masses and Couplings

The above quantization condition determines the mass spectrum of the model. The parameters we have are k_1 , k_2 , L_1 , L_2 with the restriction $k_1 < k_2 < M$ and $k_1 \sim k_2$ so that we don't introduce a new hierarchy. It is more convenient to introduce the parameters $x = k_2(L_2 - L_1)$, $w = e^{-k_1 L_1}$ and work instead with the set k_1 , k_2 , x , w . From now on we will assume that $w \ll 1$ ($w \sim \mathcal{O}(10^{-15})$) as is required if the model is to provide an explanation of the hierarchy problem).

For the region $x \gtrsim 1$ it is straightforward to show analytically that all the masses of the KK tower scale in the same way as x is varied:

$$m_n = \xi_n k_2 w e^{-x} \quad (19)$$

where ξ_n is the n -th root of $J_1(x)$. This should be compared with the KMPRS $"++"$ model in which $m_1 \propto k w e^{-2x}$ and $m_{n+1} \propto k w e^{-x}$, where x is the separation between the $"--"$ and the second $"++"$ brane. This significant difference can be explained by the fact that in the $"++"$ case the negative tension brane creates a potential barrier between the two attractive potentials created by the positive tension branes. As a result the wave function in the region of the $"--"$ brane is small due to the tunneling effect. The two attractive potentials support two bound states, one the graviton and the other the first KK mode. The mass difference

between the two is determined by the wave function in the neighbourhood of the $"-"$ brane and is thus very small. On the other hand the wave function between the two $"+"$ branes in the $"++-"$ configuration is not suppressed by the need to tunnel and hence the mass difference between the zero mode and the first KK mode is also not suppressed. This has as result the two $"+"$ branes behave approximately as one. This becomes even more clear when $x \gg 1$ where the model resembles the $"+-"$ RS construction. Indeed, in this limit the mass spectrum becomes $m_n = \xi_n k_2 e^{-k_2 L_2}$ which is exactly that of the $"+-"$ RS model with orbifold size L_2 and bulk curvature k_2 .

In the region $x \lesssim 1$ the relation of eq(19) breaks down. As reduce x the second $"+"$ comes closer and closer to the $"-"$ brane and in the limit $x \rightarrow 0$ (*i.e.* $L_2 = L_1$) the combined brane system behaves as a single $"-"$ brane, reducing to the $"+-"$ RS model. In this limit the spectrum is given by:

$$m_n = \xi_n k_1 w \quad (20)$$

which is just the one of the $"+-"$ RS model. In the region $0 \leq x \lesssim 1$ the mass spectrum interpolates between the relations (19) and (20).

The fact that there is nothing special about the first KK mode is true also for its coupling on the second $"+"$ brane. The interaction of the KK states on the second $"+"$ brane is given by:

$$\mathcal{L}_{int} = \sum_{n \geq 0} a_n h_{\mu\nu}^{(n)}(x) T_{\mu\nu}(x) \quad , \quad \text{with} \quad a_n = \left[\frac{g(z_1)}{M} \right]^{3/2} \hat{\Psi}^{(n)}(z_1) \quad (21)$$

In the RS limit ($x = 0$) all the states of the KK tower have equal coupling given by:

$$a_n = \frac{1}{w M_{Pl}} \quad (22)$$

As we increase x , the lower a state is in the tower, the more strongly it couples, *i.e.* $a_1 > a_2 > a_3 > \dots$ (with $a_1 < (w M_{Pl})^{-1}$) and tends to a constant value for high enough levels. At some point this behaviour changes, the levels cross and for $x \gtrsim 1$ the situation is reversed and the lower a state is in the tower, the more weakly it couples, *i.e.* $a_1 < a_2 < a_3 < \dots$. In this region it is possible to obtain a simple analytical expression for the couplings:

$$a_n = \frac{8\xi_n^2}{J_2(\xi_n)} \left(\frac{k_2}{k_1} \right)^{3/2} \frac{1}{w M_{Pl}} e^{-3x} \quad (23)$$

Here, we also see that the first KK state scales in exactly the same way as the remaining states in the tower with respect to x , a behaviour quite different to that in the KMPRS model in which the coupling is x -independent. Furthermore, the coupling falls as e^{-3x} , *i.e.* much faster than e^{-x} as one would naively expect. This can be explained by looking at the

origin of the x -dependence of the wavefunction. For increasing x the normalization volume coming from the region between L_1 and L_2 dominates and the normalization constant in (17) scales as $N_n \propto e^{-3x}$. This rapid decrease is not compensated by the increase of the value of the remaining wavefunction (which from (17) is approximately constant). Thus, although the two $''+$ branes in the large x limit behave as one as far as the mass spectrum is concerned, their separation actually makes the coupling of the KK modes very different.

3 The four-brane $''+--+$ Model

In this section we discuss the four-brane model in order to clarify the relation between the “bigravity” KMPRS model [8] with the GRS model[10]. In particular we wish to explore and compare the modification of gravity at large scales predicted by each model.

In the case of the KMPRS $''+--+$ model, in the limit of very large x , gravity results from the net effect of both the massless graviton and the ultralight first KK state. The modifications of gravity at very large distances come from the fact that the Yukawa type suppression of the gravitational potential coming from the KK state turns on at the Compton wavelength of the state. On the other hand, the GRS model has a continuous spectrum with no normalizable zero mode. However, the values of the KK states wavefunctions on the $''+$ brane have a “resonance”-like behaviour [16] which give rise to 4D gravity at distances smaller than the Compton wavelength of its width. Beyond this scale gravity becomes five-dimensional.

The four-brane GRS configuration can be obtained from the KMPRS model by “cutting” the $''-$ brane in half, *i.e.* instead of having one $''-$ brane one can take two $''-$ branes of half the tension of the original one ($''-1/2''$ branes), having flat spacetime between them (see Fig.(2)). Finally if the second $''+$ brane is taken to infinity together with one of the $''-1/2''$ branes we shall get precisely the GRS picture.

Let us discuss the four-brane $''+--+$ model in more detail. It consists of 5D spacetime with orbifold topology with four parallel 3-branes located at $L_0 = 0$, L_1 , L_2 and L_3 , where L_0 and L_3 are the orbifold fixed points (see Fig.(2)). The bulk cosmological constant Λ is negative (*i.e.* AdS_5 spacetime) between the branes with opposite tension and zero (*i.e.* flat spacetime) between the two $''-1/2''$ branes. The model has four parameters namely L_1 , L_2 and L_3 and Λ . For our present purposes we consider the symmetric configuration, leaving 3 parameters, l , l_- and Λ where $l \equiv L_1 = L_3 - L_2$ and $l_- \equiv L_2 - L_1$. In the absence of matter the model is described by eq(1) with

$$\Lambda(y) = \begin{cases} 0 & , y \in [L_1, L_2] \\ \Lambda & , y \in [0, L_1] \cup [L_2, L_3] \end{cases} \quad (24)$$

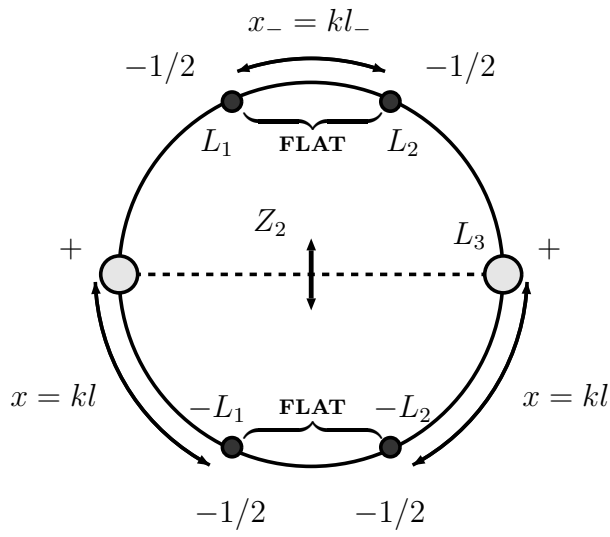


Figure 2: “+ - - +” configuration with scale equivalent “+” branes. The distance between the “+” and “- 1/2” branes is $l = L_1 = L_3 - L_2$ while the distance between the “- 1/2” branes is $l_- = L_2 - L_1$. The curvature of the bulk between the “+” and “- 1/2” branes is k .

By considering the ansatz eq(2) the “warp” function $\sigma(y)$ must satisfy:

$$\sigma'' = \sum_i \frac{V_i}{12M^3} \delta(y - L_i) \quad \text{and} \quad (\sigma')^2 = \begin{cases} 0 & , y \in [L_1, L_2] \\ k^2 & , y \in [0, L_1] \cup [L_2, L_3] \end{cases} \quad (25)$$

where $k = \sqrt{\frac{-\Lambda}{24M^3}}$ is a measure of the bulk curvature and we take $V_0 = V_3 = -2V_1 = -2V_2 \equiv V$. The solution for $y > 0$ is:

$$\sigma(y) = \begin{cases} ky & , y \in [0, L_1] \\ kL_1 & , y \in [L_1, L_2] \\ kL_1 + k(L_2 - y) & , y \in [L_2, L_3] \end{cases} \quad (26)$$

Furthermore, 4D Poincare invariance requires the fine tuned relation:

$$V = -\frac{\Lambda}{k} \quad (27)$$

In order to determine the mass spectrum and the couplings of the KK modes we consider linear “massive” metric fluctuations as in eq(8). Following the same procedure we find that the function $\hat{\Psi}^{(n)}(z)$ obeys a Schrödinger-like equation with potential $V(z)$ of the form:

$$V(z) = \frac{15k^2}{8[g(z)]^2} [\theta(z) - \theta(z - z_1) + \theta(z - z_2) - \theta(z - z_3)] - \frac{3k}{2g(z)} \left[\delta(z) - \frac{1}{2}\delta(z - z_1) - \frac{1}{2}\delta(z - z_2) + \delta(z - z_3) \right] \quad (28)$$

The conformal coordinates now are given by:

$$z \equiv \begin{cases} \frac{e^{ky}-1}{k} & , y \in [0, L_1] \\ (y-l)e^{kl} + \frac{e^{kl}-1}{k} & , y \in [L_1, L_2] \\ -\frac{1}{k}e^{2kl+kl_-}e^{-ky} + l_-e^{kl} + \frac{2}{k}e^{kl} - \frac{1}{k} & , y \in [L_2, L_3] \end{cases} \quad (29)$$

$$g(z) = \begin{cases} kz + 1 & , z \in [0, z_1] \\ kz_1 + 1 & , z \in [z_1, z_2] \\ k(z_2 - z) + kz_1 + 1 & , z \in [z_2, z_3] \end{cases} \quad (30)$$

where $z_1 = z(L_1)$, $z_2 = z(L_2)$ and $z_3 = z(L_3)$.

The potential (28) again gives rise to a massless graviton zero mode whose wavefunction is given by (16) with the same normalization convention. Note, however, that in the limit $l_- \rightarrow \infty$ this mode becomes non-normalizable (GRS case). The solution of the Schrödinger equation for the massive KK modes is:

$$\hat{\Psi}^{(n)} \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{Bmatrix} = N_n \begin{Bmatrix} \sqrt{\frac{g(z)}{k}} [Y_2(\frac{m_n}{k}g(z)) + A J_2(\frac{m_n}{k}g(z))] \\ B_1 \cos(m_n z) + B_2 \sin(m_n z) \\ \sqrt{\frac{g(z)}{k}} [C_1 Y_2(\frac{m_n}{k}g(z)) + C_2 J_2(\frac{m_n}{k}g(z))] \end{Bmatrix} \quad (31)$$

where $\mathbf{A} = [0, z_1]$, $\mathbf{B} = [z_1, z_2]$, and $\mathbf{C} = [z_2, z_3]$. We observe that the solution in the first and third interval has the same form as in the "++" model. The new feature is the second region (flat spacetime). The coefficients that appear in the solution are determined by imposing the boundary conditions and normalizing the wavefunction.

The boundary conditions (two for the continuity of the wavefunction at z_1, z_2 and four for the discontinuity of its first derivative at $0, z_1, z_2$ and z_3) result in a 6×6 homogeneous linear system which, in order to have a non-trivial solution, should have vanishing determinant. It is readily reduced to a 4×4 set of equations leading to the quantization condition:

$$\begin{vmatrix} Y_2(g_1 \frac{m}{k}) - \frac{Y_1(\frac{m}{k})}{J_1(\frac{m}{k})} J_2(g_1 \frac{m}{k}) & -\cos(mz_1) & -\sin(mz_1) & 0 \\ Y_1(g_1 \frac{m}{k}) - \frac{Y_1(\frac{m}{k})}{J_1(\frac{m}{k})} J_1(g_1 \frac{m}{k}) & \sin(mz_1) & -\cos(mz_1) & 0 \\ 0 & -\sin(mz_2) & \cos(mz_2) & Y_1(g_2 \frac{m}{k}) - \frac{Y_1(g_3 \frac{m}{k})}{J_1(g_3 \frac{m}{k})} J_1(g_2 \frac{m}{k}) \\ 0 & \cos(mz_2) & \sin(mz_2) & -Y_2(g_2 \frac{m}{k}) + \frac{Y_1(g_3 \frac{m}{k})}{J_1(g_3 \frac{m}{k})} J_2(g_2 \frac{m}{k}) \end{vmatrix} = 0 \quad (32)$$

with $g_1 = g(z_1)$, $g_2 = g(z_2)$ and $g_3 = g(z_3)$. Here we have suppressed the subscript n on the masses m_n .

3.1 The Mass Spectrum

The above quantization condition provides the mass spectrum of the model. It is convenient to introduce two dimensionless parameters, $x = kl$ and $x_- = kl_-$ (c.f. Fig.(2)) and we work from now on with the set of parameters x , x_- and k . The mass spectrum depends crucially on the distance x_- . We must recover the KMPRS spectrum in the limit $x_- \rightarrow 0$, and the GRS spectrum in the limit $x_- \rightarrow \infty$. From the quantization condition, eq(32) it is easy to verify these features and show how the "++--" spectrum smoothly interpolates between the KMPRS model and the GRS one. It turns out that the structure of the spectrum has simple x_- and x dependence in three separate regions of the parameter space:

3.1.1 The three-brane "++--" Region

For $x_- \lesssim 1$ we find that the mass spectrum is effectively x_- -independent given by the approximate form:

$$m_1 = 2\sqrt{2}ke^{-2x} \quad (33)$$

$$m_{n+1} = \xi_n ke^{-x} \quad n = 1, 2, 3, \dots \quad (34)$$

where ξ_{2i+1} is the $(i+1)$ -th root of $J_1(x)$ ($i = 0, 1, 2, \dots$) and ξ_{2i} is the i -th root of $J_2(x)$ ($i = 1, 2, 3, \dots$). As expected the mass spectrum is identical to the one of the KMPRS model for the trivial warp factor $w = 1$. The first mass is manifestly singled out from the rest of the KK tower and for large x leads to the possibility of bigravity.

3.1.2 The Saturation Region

For $1 \ll x_- \ll e^{2x}$ we find a simple dependence on x_- given by the approximate analytic form:

$$m_1 = 2k \frac{e^{-2x}}{\sqrt{x_-}} \quad (35)$$

$$m_{n+1} = n\pi k \frac{e^{-x}}{x_-} \quad n = 1, 2, 3, \dots \quad (36)$$

As x_- increases the first mass decreases less rapidly than the other levels.

3.1.3 The GRS Region

For $x_- \gg e^{2x}$ the first mass is no longer special and scales with respect to *both* x and x_- in the same way as the remaining tower:

$$m_n = n\pi k \frac{e^{-x}}{x_-} \quad n = 1, 2, 3, \dots \quad (37)$$

The mass splittings Δm tend to zero as $x_- \rightarrow \infty$ and we obtain the GRS continuum of states

The behaviour of the spectrum is illustrated in Figure 3.

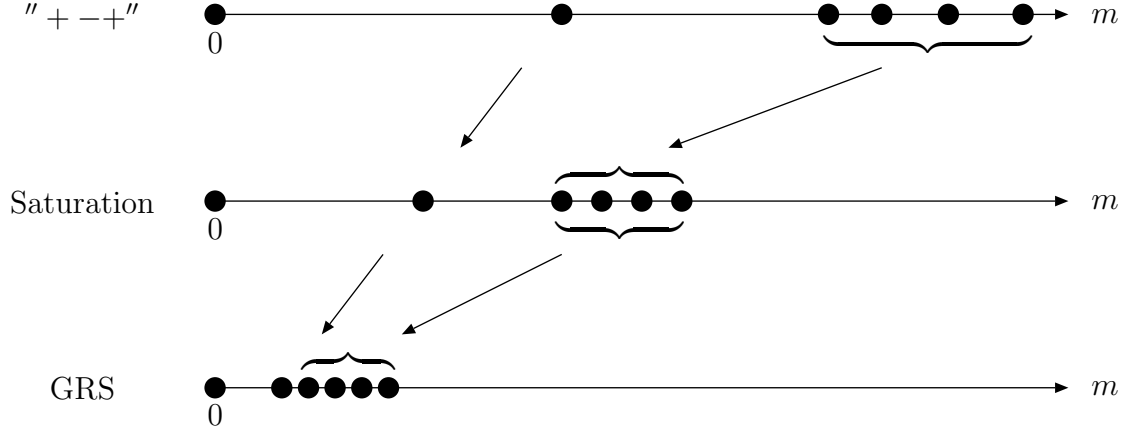


Figure 3: The behaviour of the mass of the first five KK states in the three regions of simple x , x_- dependence. The first dot at zero stands for the graviton.

3.2 Multigravity

Armed with the details how the spectrum smoothly changes between the KPMRS model ($x_- = 0$) and the GRS model ($x_- \rightarrow \infty$), we can now discuss the possibilities for modifying gravity at large distances. The couplings of the KK states with matter on the left $++$ brane are readily calculated by the interaction Lagrangian (21) with:

$$a_n = \left[\frac{g(0)}{M} \right]^{3/2} \hat{\Psi}^{(n)}(0) \quad (38)$$

3.2.1 Bigravity Region

In the KPMRS limit, $x_- \rightarrow 0$, the first KK mode has constant coupling equal to that of the 4D graviton:

$$a_1 = \frac{1}{M_*} (= a_0) \quad \text{where} \quad M_*^2 = \frac{2M^3}{k} \quad (39)$$

while the couplings of the rest of the KK tower are exponentially suppressed:

$$a_{n+1} = \frac{1}{M_*} \frac{e^{-x}}{\sqrt{J_1^2\left(\frac{m_n e^x}{k}\right) + J_2^2\left(\frac{m_n e^x}{k}\right)}} \quad n = 1, 2, 3, \dots \quad (40)$$

The gravitational potential is computed by the tree level exchange diagrams of the 4D graviton and KK states which in the Newtonian limit is:

$$V(r) = - \sum_{n=0}^{N_\Lambda} a_n^2 \frac{e^{-m_n r}}{r} \quad (41)$$

where a_n is the coupling (38) and $n = 0$ accounts for the massless graviton. The summation stops at some very high level N_Λ with mass of the order of the cutoff scale $\sim M$.

In the “bigravity” scenario, at distances $r \ll m_1^{-1}$, the first KK state and the 4D graviton contribute equally to the gravitational force, *i.e.*

$$V_{ld}(r) \approx -\frac{1}{M_*^2} \left(\frac{1}{r} + \frac{e^{-m_1 r}}{r} \right) \approx -\frac{G_N}{r} \quad (42)$$

where $G_N \equiv \frac{2}{M_*^2}$. For distances $r \gtrsim m_1^{-1}$ the Yukawa suppression effectively reduces gravity to half its strength. Astronomical constraints and the requirement of the observability of this effect demand that for $k \sim M_{\text{Pl}}$ we should have x in the region 65-70. Moreover, at distances $r \lesssim m_2^{-1}$ the Yukawa interactions of the remaining KK states are significant and will give rise to a short distance correction. This can be evaluated by using the asymptotic expression of the Bessel functions in (40) since we are dealing with large x and summing over a very dense spectrum, giving:

$$V_{sd}(r) = -\frac{G_N}{k} \sum_{n=2}^{N_\Lambda} \frac{k\pi}{2e^x} \frac{m_n}{2k} \frac{e^{-m_n r}}{r} \quad (43)$$

At this point we exploit the fact that the spectrum is nearly continuum above m_2 and turn the sum to an integral with the first factor in (50) being the integration measure, *i.e.* $\sum \frac{k\pi}{2e^x} = \sum \Delta m \rightarrow \int dm$ (this follows from eq(34) for the asymptotic values of the Bessel roots). Moreover, we can extend the integration to infinity because, due to the exponential suppression of the integrand, the integral saturates very quickly and thus the integration over the region of very large masses is irrelevant. The resulting potential is now:

$$V_{sd}(r) = -\frac{G_N}{k} \int_{m_2}^{\infty} dm \frac{m}{2k} \frac{e^{-m r}}{r} \quad (44)$$

The integration is easily performed and gives:

$$V_{sd}(r) \simeq -\frac{G_N}{2r} \frac{1 + m_2 r}{(kr)^2} e^{-m_2 r} \quad (45)$$

We see these short distance corrections are significant only at Planck scale lengths $\sim k^{-1}$.

3.2.2 The GRS Region

In the GRS limit, $x_- \gg e^{2x}$, we should reproduce the “resonance”-like behaviour of the coupling in the GRS model. In the following we shall see that indeed this is the case and we will calculate the first order correction to the GRS potential for the case x_- is large but finite.

For the rest of the section we split the wavefunction (31) in two parts, namely the normalization N_n and the unnormalized wavefunction $\tilde{\Psi}^{(n)}(z)$, *i.e.* $\hat{\Psi}^{(n)}(z) = N_n \tilde{\Psi}^{(n)}(z)$. The former is as usual chosen so that we get a canonically normalized Pauli-Fierz Lagrangian for the 4D KK modes $h_{\mu\nu}^{(n)}$ and is given by:

$$N_n^2 = \frac{1/2}{2 \int_0^{z_1} dz \left[\tilde{\Psi}^{(n)}(z) \right]^2 + \int_{z_1}^{z_2} dz \left[\tilde{\Psi}^{(n)}(z) \right]^2} \quad (46)$$

The value of $\tilde{\Psi}^{(n)}(z)$ on the left “+” brane is, for $m_n \ll k$:

$$\tilde{\Psi}_{(n)}^2(0) \simeq \frac{16k^3}{\pi^2 m_n^4} \quad (47)$$

It is convenient to split the gravitational potential given by the relation (41) into two parts:

$$V(r) = -\frac{1}{M^3} \sum_{n=1}^{N_{x_-}-1} \frac{e^{-m_n r}}{r} N_n^2 \tilde{\Psi}_{(n)}^2(0) - \frac{1}{M^3} \sum_{n=N_{x_-}}^{N_\Lambda} \frac{e^{-m_n r}}{r} N_n^2 \tilde{\Psi}_{(n)}^2(0) \quad (48)$$

As we shall see this separation is useful because the first N_{x_-} states give rise to the long distance gravitational potential V_{ld} while the remaining ones will only contribute to the short distance corrections V_{sd} .

- **Short Distance Corrections**

We first consider the second term. The normalization constant in this region is computed by considering the asymptotic expansions of the Bessel functions with argument $\frac{g(z_1)m_n}{k}$. It is easily calculated to be:

$$N_n^2 = \frac{\pi^3 m_n^5}{32k^3 g(z_1)x_-} \left[\frac{1}{1 + \frac{2}{x_-}} \right] \quad (49)$$

If we combine the above normalization with unnormalized wavefunction (47), we find

$$V_{sd}(r) \simeq -\frac{1}{M^3} \sum_{n=N_{x_-}}^{N_\Lambda} \frac{k\pi}{x_- e^x} \frac{m_n}{2k} \frac{e^{-m_n r}}{r} \left[\frac{1}{1 + \frac{2}{x_-}} \right] \quad (50)$$

Since we are taking $x_- \gg e^{2x}$, the spectrum tends to continuum, *i.e.* $N_n \rightarrow N(m)$, $\tilde{\Psi}_{(n)}(0) \rightarrow \tilde{\Psi}(m)$, and the sum turns to an integral where the first factor in (50) is the integration measure, *i.e.* $\sum \frac{k\pi}{x_- e^x} = \sum \Delta m \rightarrow \int dm$ (*c.f.* eq(37)). Moreover, as before we can again extend the integration to infinity. Finally, we expand the fraction involving x_- keeping the first term in the power series to obtain the potential:

$$V_{sd}(r) \simeq -\frac{1}{M^3} \int_{m_0}^{\infty} dm \frac{e^{-mr}}{r} \frac{m}{2k} \left(1 - \frac{2}{x_-}\right) \quad (51)$$

where $m_0 = ke^{-x}$. The integral is easily calculated and the potential reads:

$$V_{sd}(r) \simeq -\frac{G_N}{2r} \frac{1+m_0 r}{(kr)^2} \left(1 - \frac{2}{x_-}\right) e^{-m_0 r} \quad (52)$$

where we identified $G_N \equiv \frac{k}{M^3}$ for reasons to be seen later. The second part of the above potential is the first correction coming from the fact that x_- is finite. Obviously this correction vanishes when $x_- \rightarrow \infty$. Note that the above potential gives corrections to the Newton's law only at distances comparable to the Planck length scale.

• Multigravity: 4D and 5D gravity

We turn now to the more interesting first summation in eq(48) in order to show that the coupling indeed has the “resonance”-like behaviour for $\Delta m \rightarrow 0$ responsible for 4D Newtonian gravity at intermediate distances and the 5D gravitational law for cosmological distances. This summation includes the KK states from the graviton zero mode up to the N_{x_-} -th level. The normalization constant in this region is computed by considering the series expansion of all the Bessel functions involved. It is easily calculated to be:

$$N_n^2 \simeq \frac{\pi^2 m_n^4}{4g(z_1)^4 x_-} \left[\frac{1}{m_n^2 + \frac{\Gamma^2}{4} + \frac{8k^2}{g(z_1)^4 x_-}} \right] \quad (53)$$

where $\Gamma = 4ke^{-3x}$. If we combine the above normalization with the unnormalized wave-function (47), we find that the long distance gravitational potential is:

$$V_{ld}(r) = -\frac{1}{M^3} \sum_{n=0}^{N_{x_-}} \frac{\pi k}{x_- e^x} \frac{4k^2}{\pi g(z_1)^3} \frac{e^{-m_n r}}{r} \left[\frac{1}{m_n^2 + \frac{\Gamma^2}{4} + \frac{8k^2}{g(z_1)^4 x_-}} \right] \quad (54)$$

Again, since we are taking $x_- \gg e^{2x}$, the above sum will turn to an integral with $\sum \Delta m \rightarrow \int dm$. Moreover, we can safely extend the integration to infinity since the integral saturates very fast for $m \lesssim \Gamma/4 \equiv r_c^{-1} \ll ke^x$. If we also expand the fraction in brackets keeping the first term in the power series, we find the potential:

$$\begin{aligned} V_{ld}(r) \simeq & -\frac{1}{M^3} \int_0^{\infty} dm \frac{4k^2}{\pi g(z_1)^3} \frac{e^{-mr}}{r} \frac{1}{m^2 + \frac{\Gamma^2}{4}} \\ & + \frac{1}{M^3} \int_0^{\infty} dm \frac{32k^4}{x_- \pi g(z_1)^7} \frac{e^{-mr}}{r} \frac{1}{(m^2 + \frac{\Gamma^2}{4})^2} \end{aligned} \quad (55)$$

The first part is the same as in the GRS model potential, whereas the second one is the first correction that comes from the fact that x_- is still finite though very large. Note that the width of the “resonance” scales like e^{-3x} , something that is compatible with the scaling law of the masses ($m_n = n\pi k \frac{e^{-x}}{x_-}$), since we are working at the region where $x_- \gg e^{2x}$, *i.e.* $m_n \ll n\pi k e^{-3x}$. The above integrals can be easily calculated in two interesting limits.

For $k^{-1} \ll r \ll r_c$ the potential is given approximately by :

$$V_{ld}(r \ll r_c) \simeq -\frac{G_N}{r} \left(1 - \frac{e^{2x}}{x_-}\right) \quad (56)$$

where we have identified $G_N \equiv \frac{k}{M^3}$ to obtain the normal 4D Newtonian potential. Note that since $x_- \gg e^{2x}$, the $1/x_-$ term is indeed a small correction.

In the other limit, $r \gg r_c$, the integrand is only significant for values of m for which the m^2 term in the denominator of the “Breit-Wigner” can be dropped and the potential becomes:

$$V_{ld}(r \gg r_c) \simeq -\frac{G_N r_c}{\pi r^2} \left(1 - \frac{2e^{2x}}{x_-}\right) \quad (57)$$

The fact that Newtonian gravity has been tested close to the present horizon size require that for $k \sim M_{\text{Pl}}$ we should have $x \gtrsim 45$ -50.

Finally we note that if we take the $x_- \rightarrow \infty$ we recover the GRS result

$$\lim_{x_- \rightarrow \infty} V_{+---+}(r, x_-) = V_{GRS}(r) \quad (58)$$

• Back to 4D gravity

As we have just seen, probing larger distances than r_c , the 4D gravitational potential changes to a 5D one. This is the most significant characteristic of the GRS model. In the case that x_- is large compared to e^{2x} but still finite, there is another distinct region of interest, namely $r \gg m_1^{-1}$. This follows from the fact that, in this limit, the spectrum is still discrete. For distances larger than of the order of the corresponding wavelength of the first KK mode, the contribution to gravity from the KK tower is suppressed and thus the zero mode gives the dominant contribution, leading to the 4D Newtonian potential again. In this case the strength of the gravitational interaction is a small fraction of the strength of the intermediate 4D gravity. More precisely, the contribution of the massless graviton is $\frac{1}{x_-}$ suppressed and thus vanishes when $x_- \rightarrow \infty$, something that is expected since in this limit there is no normalizable zero mode. The gravitational potential in this case is:

$$V_{4D}(r) = -\frac{1}{M^3} \frac{1}{r} N_0^2 \tilde{\Psi}_{(0)}^2(0) = -\frac{G_N}{r} \frac{e^{2x}}{x_-} \quad (59)$$

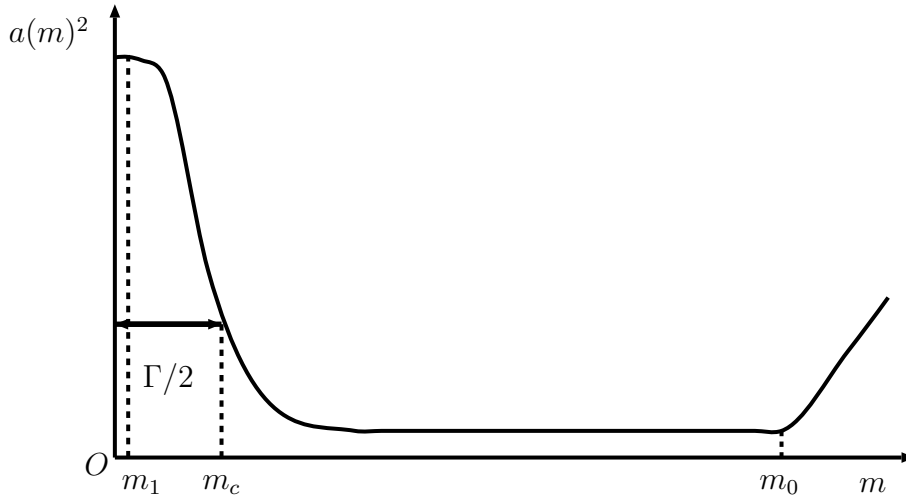


Figure 4: The behaviour of the coupling, $a(m)$, in the limit of $x_- \gg e^{2x}$. Three regions of interest are indicated. The region $m > m_0$ gives rise to short distance corrections. The $m_1 \ll m \ll m_c$ region gives rise to 4D gravity at intermediate distances and 5D gravity at ultra large distances. For distances $r \gg m_1^{-1}$, the zero mode gives the dominant contribution and thus we return to 4D gravity.

Obviously this 4D region disappears at the limit $x_- \rightarrow \infty$ since the spectrum becomes continuum and thus the 5D gravity “window” extends to infinity. We should note finally that for the values of x that we consider here this final modification of gravity occurs at distances well above the present horizon.

4 The Crystal Universe Model

The last model we consider is the crystal universe introduced in [12] (see also [13]). It consists of an infinite array of parallel 3-branes in a 5D AdS space with cosmological constant Λ . For simplicity we assume that the branes are equidistant with distance, l , between two successive branes. Needless to say, in this case all the “+” branes have warp factor $w = 1$ with respect to a “+” brane sitting at the origin of the 5-th dimension coordinate. The metric ansatz that has 4D Poincaré invariance is again given by eq(2) where the $\sigma(y)$ function is constrained by the the Einstein equations to have the sawtooth form:

$$\sigma(y) = k \sum_{j=-\infty}^{+\infty} (y - 2jl) [2\theta(y - 2jl) - \theta(y - (2j - 1)l) - \theta(y - (2j + 1)l)] \quad (60)$$

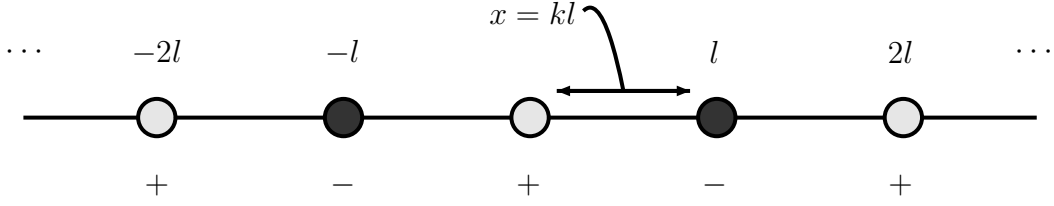


Figure 5: The Crystal Universe made up of an infinite array of "+" and "-" branes with lattice spacing l and bulk curvature k .

The tensions of successive branes are required to be opposite and equal to $\pm\Lambda/k$, where $k = \sqrt{\frac{-\Lambda}{24M^3}}$ is a measure of the curvature of the bulk and M the 5D fundamental scale.

We consider the general fluctuations around the previous vacuum ansatz of the form (8) and decompose as usual the perturbation tensor $h_{\mu\nu}(x, y)$ as:

$$h_{\mu\nu}(x, y) = \int dm h_{\mu\nu}(m, x) \Psi(m, y) \quad (61)$$

where $\Psi(m, y)$ is a complex function that arises from the linear combination of the independent graviton polarizations. This leads to the Schrödinger equation (10) for the wave function $\hat{\Psi}(m, z) \equiv \Psi(m, y)e^{\sigma/2}$ with potential:

$$V(z) = \frac{15k^2}{8[g(z)]^2} - \frac{3k}{2g(z)} \sum_{j=-\infty}^{+\infty} (-)^j \delta(z - z_j) \quad (62)$$

The z coordinates have been defined as usual to be the ones that make the background metric conformally flat. In these coordinates the branes sit at the points $z_j = j \frac{e^{kl}-1}{k} \equiv jz_l$ and the function $g(z)$ is:

$$g(z) = 1 + k \sum_{j=-\infty}^{+\infty} (z - z_{2j}) [2\theta(z - z_{2j}) - \theta(z - z_{2j-1}) - \theta(z - z_{2j+1})] \quad (63)$$

The solution of the Schrödinger equation for the wavefunctions for two adjacent cells is given[12] in terms of Hankel functions:

$$\hat{\Psi}(m, z) = N_m \sqrt{\frac{g(z)}{k}} \left[\left\{ \begin{array}{c} 1 \\ B \\ e^{2iqz_l} \\ e^{2iqz_l} B \end{array} \right\} H_2^+ \left(\frac{m}{k} g(z) \right) + \left\{ \begin{array}{c} A \\ C \\ e^{2iqz_l} A \\ e^{2iqz_l} C \end{array} \right\} H_2^- \left(\frac{m}{k} g(z) \right) \right] \quad (64)$$

in the regions $z \in [0, z_l]$, $z \in [z_l, 2z_l]$, $z \in [2z_l, 3z_l]$ and $z \in [3z_l, 4z_l]$ respectively ⁷. In

⁷We use as [12] Hankel instead of real Bessel functions in order to encode the gravitons phase rotation.

the above expression N_m is an overall normalization constant, q is the Bloch wave quasi-momentum and the constants A, B, C are given by:

$$A = \frac{(h_1^+ h_2^- + h_1^- h_2^+) \hat{h}_2^+ - (h_1^- h_2^+ - h_1^+ h_2^-) e^{2iqz_l} \hat{h}_2^+ - 2h_1^+ h_2^+ \hat{h}_2^-}{(h_1^+ h_2^- + h_1^- h_2^+) \hat{h}_2^- + (h_1^- h_2^+ - h_1^+ h_2^-) e^{2iqz_l} \hat{h}_2^- - 2h_1^- h_2^- \hat{h}_2^+} \quad (65)$$

$$B = \frac{h_1^+ h_2^- + h_1^- h_2^+ + 2h_1^- h_2^- A}{h_1^- h_2^+ - h_1^+ h_2^-} \quad (66)$$

$$C = \frac{2h_1^+ h_2^+ + (h_1^+ h_2^- + h_1^- h_2^+) A}{h_1^+ h_2^- - h_1^- h_2^+} \quad (67)$$

where $h_n^\pm \equiv H_n^\pm \left(\frac{m}{k} g(z_l) \right)$ and $\hat{h}_n^\pm \equiv H_n^\pm \left(\frac{m}{k} \right)$. The above coefficients of the Hankel functions were determined by the boundary conditions, *i.e.* continuity of the wavefunction at z_l , $2z_l$, $3z_l$, discontinuity of its first derivative at z_l , $3z_l$ and the Bloch wave conditions relating the wavefunction at the edges of each cell. The last remaining boundary condition of the discontinuity of the first derivative of the wavefunction at $2z_l$ gives us the band equation connecting q and m . We disagree with the relation given in [12] and instead we find:

$$\cos(2qz_l) = \frac{(j_2 y_1 + j_1 y_2)(\hat{j}_2 \hat{y}_1 + \hat{j}_1 \hat{y}_2) - 2\hat{j}_1 \hat{j}_2 y_1 y_2 - 2j_1 j_2 \hat{y}_1 \hat{y}_2}{(j_2 y_1 - j_1 y_2)(\hat{j}_2 \hat{y}_1 - \hat{j}_1 \hat{y}_2)} \equiv f(m) \quad (68)$$

where again $j_n \equiv J_n \left(\frac{m}{k} g(z_l) \right)$, $y_n \equiv Y_n \left(\frac{m}{k} g(z_l) \right)$ and $\hat{j}_n \equiv J_n \left(\frac{m}{k} \right)$, $\hat{y}_n \equiv Y_n \left(\frac{m}{k} \right)$. This dispersion relation *always* gives a band at zero as is to be expected intuitively. Defining the parameter $x \equiv kl$, we see that for $x \gtrsim 10$ we may reliably approximate the width of the zero mode band by:

$$\Gamma_0 = 2\sqrt{2}ke^{-2x} \quad (69)$$

while the separation of the zeroth and the first band is

$$\Delta\Gamma_1 = \xi_1 ke^{-x} \quad (70)$$

with ξ_1 the 1st root of $J_1(x)$. This characteristic behaviour of the above widths with respect to x implies that we can have a viable multigravity scenario if $\Gamma_0^{-1} \gtrsim 10^{26} \text{cm}$ and at the same time $\Delta\Gamma_1^{-1} \lesssim 1 \text{mm}$. This can happen for $k \sim M_{\text{Pl}}$ and $x \approx 68$.

The band structure has the following form. The first bands have very narrow widths Γ_i with spacing $\Delta\Gamma_i$ between them (*i.e.* between the i -th and the $(i-1)$ -th band), which are approximately:

$$\Gamma_i \approx \epsilon_i \xi_i^2 k e^{-3x} \quad (71)$$

$$\Delta\Gamma_i \approx (\xi_i - \xi_{i-1}) k e^{-x} \quad (72)$$

where ξ_{2n+1} is the $(n+1)$ -th root of $J_1(x)$, ξ_{2n} is the n -th root of $J_2(x)$, and $\epsilon_{2n+1} = \frac{\pi}{4} \frac{Y_1(\xi_{2n+1})}{J_2(\xi_{2n+1})}$, $\epsilon_{2n} = \frac{\pi}{4} \frac{Y_2(\xi_{2n})}{J_1(\xi_{2n})}$. As we move on to higher bands their width is increasing and the spacing

between them is decreasing. In the limit $m \rightarrow \infty$, the spacing disappears, *i.e.* $\Delta\Gamma_i \rightarrow 0$, while their width tends to $\Gamma_i \rightarrow \frac{\pi}{2}ke^{-x}$. In this limit the function $f(m)$ in (68) tends to $f(m) = \cos(2z_l m)$ and we have no forbidden zones.

In order to find the gravitational potential we should at first find how the KK modes couple to matter and determine their spectral density. Instead of working with wavefunctions in an infinite extra dimension with continuous normalization, it is more convenient effectively to compactify the system to a circle of length $2\mathcal{N}l$, by assuming that we have a finite crystal of \mathcal{N} cells $[2jl, 2(j+1)l]$. We shall find the gravitational potential is independent of \mathcal{N} , so we will be able to take the limit $\mathcal{N} \rightarrow \infty$ and decompactify the geometry.

In this regularization scheme we can readily see that the ever-present zero mode, corresponding to the 4D graviton, has the wavefunction:

$$\hat{\Psi}(0, z) = \frac{\sqrt{k/\mathcal{N}}}{[g(z)]^{3/2}} \quad (73)$$

and its coupling to matter on the central positive brane is:

$$a(0) = \frac{1}{\sqrt{\mathcal{N}}} \sqrt{\frac{k}{M^3}} \equiv \frac{1}{\sqrt{\mathcal{N}}} \frac{1}{M_*} \quad (74)$$

The wavefunction eq(73) is a smooth limit of eq(64) for $m \rightarrow 0$. Indeed, for large x we find the normalization constant in (64) in the region of the zero mode band is approximately:

$$N_m = \frac{\sqrt{2}}{z_l^2 m} \frac{1}{\sqrt{\mathcal{N}}} \quad (75)$$

The approximation of the Hankel functions with their first term of their power series proves our previous claim. Furthermore, we can see that the KK states in the whole zero mode band for large x will couple to matter in the central positive brane as:

$$a(m) = \frac{1}{\sqrt{\mathcal{N}}} \frac{1}{M_*} e^{i\phi(m)} \quad (76)$$

with $\phi(m)$ an unimportant m -dependent phase that doesn't appear in observable quantities.

In the limit $m \rightarrow \infty$ the Hankel functions can be approximated by their asymptotic form. The wavefunction is then almost constant in z -space and the KK states couple to matter on the central positive brane as:

$$a(m) = \frac{1}{\sqrt{\mathcal{N}}} \frac{1}{\sqrt{2}e^x M_*} e^{i\phi(m)} \quad (77)$$

where again $\phi(m)$ is an unimportant m -dependent phase. The couplings in the low lying bands interpolate between the above limiting cases.

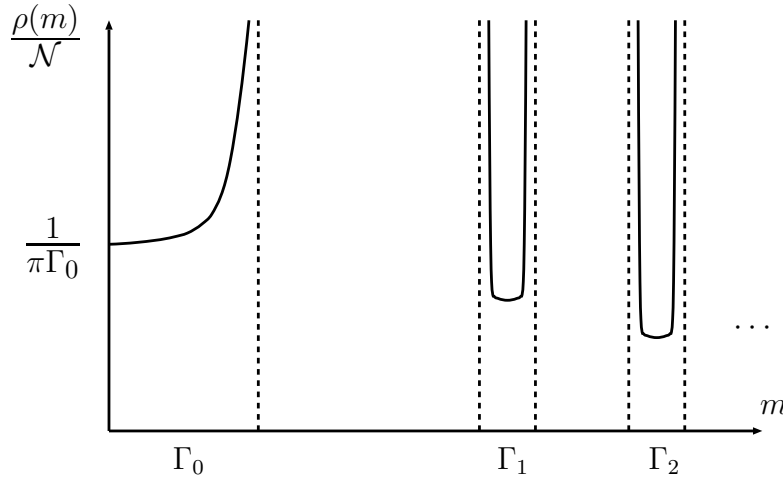


Figure 6: Spectral density for the first three bands

The standard procedure to calculate the spectral density is at first to find the number of states with masses smaller than m and then differentiate with respect to m . In our regularization prescription it is straightforward to count the states in the quasi-momentum space with masses lighter than m . The latter is simply:

$$N_{tot}(m) = \frac{\mathcal{N}z_l}{\pi}q \quad (78)$$

so the spectral density is its derivative:

$$\rho(m) = \frac{dN_{tot}(m)}{dm} = \frac{\mathcal{N}z_l}{\pi} \frac{dq}{dm} = (\mp) \frac{\mathcal{N}}{2\pi} \frac{1}{\sqrt{1-f(m)^2}} \frac{df(m)}{dm} \quad (79)$$

where the plus and minus signs will succeed each other for neighboring bands, starting with minus for the zero mode band. For the zero band we can have a reliable approximation for large x of the above function:

$$\rho(m) = \mathcal{N} \frac{1}{\pi\Gamma_0} \frac{1}{\sqrt{1-(m/\Gamma_0)^2}} \quad (80)$$

Thus the spectral density diverges as $m \rightarrow \Gamma_0$ but the divergence is integrable and doesn't cause any problem to the following calculations. Actually, the spectral density diverges at the edges of every band since at these points $\sqrt{1-f(m)^2} \rightarrow 0$ while $\frac{df(m)}{dm} \neq 0$. The only point that this doesn't happen is at $m = 0$ where $\frac{df(m)}{dm} \rightarrow 0$ in such a way that the result is finite.

In the limit $m \rightarrow \infty$ the bands disappear and the spectrum has constant spectral density:

$$\rho(m) = \mathcal{N} \frac{e^x}{k\pi} \quad (81)$$

The gravitational potential, taking into account the Yukawa suppressions of the KK states, is simply:

$$V(r) = - \int_0^\infty dm |a(m)|^2 \rho(m) \frac{e^{-mr}}{r} \quad (82)$$

For distances $r \gg \Delta\Gamma_1^{-1}$ we have the effective potential:

$$\begin{aligned} V(r) &= -\frac{1}{M_*^2} \frac{1}{r} \int_0^{\Gamma_0} dm \frac{\rho(m)}{\mathcal{N}} e^{-mr} = -\frac{1}{\pi M_*^2} \frac{1}{r} \int_0^1 d\xi \frac{e^{-(\Gamma_0 r)\xi}}{\sqrt{1-\xi^2}} \\ &= -\frac{1}{2M_*^2} \frac{1}{r} [I_0(\Gamma_0 r) - L_0(\Gamma_0 r)] \end{aligned} \quad (83)$$

where I_0 is the 0-th modified Bessel function and L_0 is the 0-th modified Struve function. The gravitational potential is \mathcal{N} -independent as expected. For distances $\Delta\Gamma_1^{-1} \ll r \ll \Gamma_0^{-1}$ the above functions tend to $I_0(\Gamma_0 r) \rightarrow 1$ and $L_0(\Gamma_0 r) \rightarrow 0$, so we recover the 4D Newton law with $G_N = \frac{1}{2M_*^2}$:

$$V(r \ll \Gamma_0^{-1}) = -\frac{G_N}{r} \quad (84)$$

On the other hand, for distances $r \gg \Gamma_0^{-1}$ the asymptotic expansion of the difference of the modified Bessel and Struve functions gives a 5D Newton law:

$$V(r \gg \Gamma_0^{-1}) = -\frac{2G_N}{\pi\Gamma_0} \frac{1}{r^2} \quad (85)$$

In case we take the crystal to be finite, there will appear a region for $r \gg m_1^{-1}$ where gravity will turn again to 4D. However, as explained in the previous section, this region is well above the universe horizon and thus of no phenomenological interest.

Finally, for distances $r \lesssim \Delta\Gamma_1^{-1}$ we will start to feel the short distance Yukawa type modifications to gravity due to the presence of the bands above the zeroth one. As in the zero hierarchy “bigravity” and “+ − − +” “multigravity” model we expect these corrections to be important at scales of order the Planck length.

5 Discussion and Conclusions

We have shown that in models in which negative tension branes are sandwiched between positive tension branes there are anomalously light KK states that can lead to modifications of gravity. So far we have implicitly assumed that the gravitational force generated by a massive graviton is identical to that generated by a massless one but this is not, in general, the case because the massive graviton has additional degrees of freedom which do not decouple in the massless limit. As a result the interaction generated by a massive graviton

violates the normal 4D relation between the gravitational interactions of matter and light. This may be seen explicitly from the form of the massive and massless graviton propagators. Up to terms involving momentum vectors the propagator has the form[19]

$$G^{\mu\nu,\alpha\beta}(x-x') = \int \frac{dp^4}{(2\pi)^4} \frac{1/2 (g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta}) - tg^{\mu\nu}g^{\alpha\beta}}{p^2 - m^2 - i\epsilon} e^{-ip(x-x')} \quad (86)$$

where for $m \neq 0$, $t = 1/3$ but for $m = 0$, $t = 1/2$ (see [17, 18]). The difference between the two propagators

$$\delta G^{\mu\nu,\alpha\beta}(x-x') = \frac{1}{6} \int \frac{dp^4}{(2\pi)^4} \frac{g^{\mu\nu}g^{\alpha\beta}}{p^2 - m^2 - i\epsilon} e^{-ip(x-x')}$$

is due to the additional helicity components needed for a massive graviton. There are three additional components needed. Two, corresponding to the graviphotons, decouple at low energies as they are derivatively coupled to the conserved energy-momentum tensor. The third, corresponding to a scalar component, does not decouple and is responsible for δG . The contribution of δG to the one-graviton exchange amplitude for any two four-dimensional sources $T_{\mu\nu}^1$ and $T_{\alpha\beta}^2$ vanishes for light for which $T = T_\mu^\mu = 0$ but not for matter because $T = T_0^0 \neq 0$, hence the normal 4D gravity relation between the interactions of matter and light is violated. This is inconsistent with observation, for example the bending of light by the sun.

It has been argued that this violation does *not* occur in the models discussed above. Two mechanisms have been identified. The first [19] involves a cancellation of the contribution from δG due to an additional scalar field, the radion, the moduli field associated with the size of the new dimension. This field must have unusual properties because it is known that a normal scalar contributes with the same sign as δG and therefore cannot cancel it. An explicit computation of Pilo et al [19] of the properties of the radion in the GRS model show that it indeed has a negative kinetic energy term leading to a contribution which cancels the troublesome δG contribution. As a result the theory indeed gives rise to 4D gravity at intermediate distances. The presence of the ghost radion field is worrying for the consistency of the theory for it gives rise to “antigravity” at extremely large scales when the theory become five dimensional. Moreover as the ghost energy is unbounded from below it is likely that any theory is unstable when coupled to such a state. It has been suggested that the problems associated with the ghost apply to all theories in which there is a negative tension brane sandwiched between positive tension branes because such theories violate the weak energy condition[20]. The latter has been shown[21] to require $\sigma''(y) \geq 0$ and, as we have seen, models which have intermediate negative tension branes violate this condition. However it is not clear that theories that violate the weak energy condition are unacceptable[23]. Indeed it has been demonstrated that any theory involving a scalar field necessarily violates the condition[24] and the condition also does not apply to the gravitational sector [25]. Certainly more work is needed to clarify these issues.

The second mechanism capable of cancelling the δG contribution was identified in [11] offers some hope that the problems associated with the ghost state may be avoided. The mechanism follows from the underlying 5D structure of the theory. The graviton propagator in five dimensions has the tensor structure [18]

$$G^{MN,PQ} \sim [1/2 (g^{MP} g^{NQ} + g^{MQ} g^{NP}) - t g^{MN} g^{PQ}] + O(pp/p^2) \quad (87)$$

where $t = 1/3$ i.e. the same as for the *massive* four-dimensional graviton. The full five-dimensional one graviton exchange amplitude has the form

$$T_{MN}^1 G^{MN,PQ} T_{PQ}^2 \propto T_{MN}^1 [1/2 (g^{MP} g^{NQ} + g^{MQ} g^{NP}) - t g^{MN} g^{PQ}] T_{PQ}^2 \quad (88)$$

where T_{MN} is the five dimensional stress-energy tensor. In the compactifications considered here $T_{\mu 5}$ vanishes but in general T_{55} does not. Thus the total amplitude can be written as

$$\begin{aligned} T_{\mu\nu}^1 T^{2\mu\nu} - t T^1 T^2 + (1-t) T_5^{15} T_5^{25} - t (T_5^{15} T^2 + T^1 T_5^{25}) &= \left[T_{\mu\nu}^1 T^{2\mu\nu} - \frac{1}{2} T^1 T^2 \right] \\ &+ (1/2 - t) T^1 T^2 + (1-t) T_5^{15} T_5^{25} - t (T_5^{15} T^2 + T^1 T_5^{25}) \end{aligned} \quad (89)$$

The first term in square brackets is the normal four-dimensional amplitude corresponding to massless graviton exchange. The second term takes account of the δG contribution as well as the contribution from T_5^5 . It may be written in the form

$$\frac{1}{6} (T^1 - 2T_5^{15}) (T^2 - 2T_5^{25}) \quad (90)$$

In [11] it was pointed out that this contribution necessarily vanishes for stable brane configurations. To see this we first note that all observable amplitudes involving states confined to the visible sector brane have zero momentum transfer in the fifth direction. This means that we have to integrate over the fifth coordinate $\int dy \sqrt{-G^{(5)}} (T - 2T_5^5)$. However it was shown in [22] that this combination vanishes in any theory which stabilises the extra dimension. As a result the contribution of the second line of eq(89) corresponding to δG vanishes and one is left with the usual four dimensional gravitational interaction. Note that this explanation does not require a field with negative kinetic energy as it arranges for the additional (scalar) graviton component associated with a massive graviton to decouple from matter via a cancellation of the various contributions to the stress energy tensor.

As discussed above there may still be a troublesome negative kinetic energy radion field but now it too decouples because it also couples to $\int dy \sqrt{-G^{(5)}} (T - 2T_5^5)$. This may eliminate the problems associated with a negative kinetic energy scalar. Indeed it suggests that we should be able to introduce an additional field to cancel the ghost field without disturbing the 4D structure of the gravitational interactions.

In summary, in this paper we discussed three models based on different brane configurations, the three-brane $"++-"$ model, the four-brane $"+-+ "$ “multigravity” model and the crystalline “multigravity” model. The $"++-"$ model provides an instructive example showing that we cannot have special light KK states (and thus “multigravity”) without intermediate $"-"$ branes. This model, however, does provide a way of having the visible sector on a $"+"$ brane with an hierarchical “warp” factor. The second $"+-+ "$ “multigravity” model provides an interpolation between the KMPRS and the GRS models and leads to a new possibility in which gravity changes from 4D to 5D and then back again to 4D. The third model, the infinite crystalline “multigravity” model provides a further example of how the “multigravity” scenario appears every time that we have $"-"$ branes between $"+"$ branes. Again gravity changes from 4D to 5D at ultralarge distances (and back to 4D if the crystal is finite). While models with intermediate negative tension branes have very interesting phenomenological implications there is a question whether they are consistent as they necessarily violate the weak energy condition[20, 21]. Related to this is the fact that they may have ghost states with negative kinetic energy. We have shown that if the brane configurations are stabilised these ghost states decouple, offering the possibility that the problems associated with such states may be avoided. Moreover the same decoupling condition ensures that the gravitational interactions associated with the KK excitations of gravity responsible for “multigravity” couple in the usual 4D way to matter and radiation.

Acknowledgments: We would like to thank N. Mavromatos for usefull discussions. S.M.’s work is supported by the Hellenic State Scholarship Foundation (IKY) No. 8117781027. A.P.’s work is supported by the Hellenic State Scholarship Foundation (IKY) No. 8017711802. The work of I.I.K. and G.G.R. is supported in part by PPARC rolling grant PPA/G/O/1998/00567, the EC TMR grant FMRX-CT-96-0090 and by the INTAS grant RFBR - 950567.

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